Gaps Project

Multiples of irrational numbers

$[\![(n+1)\sqrt{2}]\!] - [\![n\sqrt{2}]\!]$ for integers n≥0.

Prove that a_n =1 or 2 for all n > 0.

By expanding $[\![n\sqrt{2}+\sqrt{2}]\!]-[\![n\sqrt{2}]\!]$ We have $[\![(n+1)\sqrt{2}]\!]-[\![n\sqrt{2}]\!]$

Since $\sqrt{1}<\sqrt{2}<\sqrt{4}$, then $1<\sqrt{2}<2$

$1+n\sqrt{2}<\sqrt{2}+n\sqrt{2}<2+n\sqrt{2}$ so

$[\![1+n\sqrt{2}]\!]\leq[\![\sqrt{2}+n\sqrt{2}]\!]\leq[\![2+n\sqrt{2}]\!]$. We can note

$$[\![2+n\sqrt{2}]\!]-[\![1+n\sqrt{2}]\!]=2+[\![n\sqrt{2}]\!]-(1+[\![n\sqrt{2}]\!])=2+[\![n\sqrt{2}]\!]-1-[\![n\sqrt{2}]\!]=2-1$$
$$=1$$

$[\![\sqrt{2}+n\sqrt{2}]\!]\in\mathbb{Z}$

$[\![1+n\sqrt{2}]\!]\leq[\![\sqrt{2}+n\sqrt{2}]\!]\leq[\![2+n\sqrt{2}]\!]$

Therefore $[\![\sqrt{2}+n\sqrt{2}]\!]=[\![1+n\sqrt{2}]\!]$ or $[\![\sqrt{2}+n\sqrt{2}]\!]=[\![2+n\sqrt{2}]\!]$

So $[\![(n+1)\sqrt{2}]\!]=[\![1+n\sqrt{2}]\!]$ or $[\![(n+1)\sqrt{2}]\!]=[\![2+n\sqrt{2}]\!]$

Therefore

$a_n=[\![(n+1)\sqrt{2}]\!]-[\![n\sqrt{2}]\!]=[\![1+n\sqrt{2}]\!]-[\![n\sqrt{2}]\!]=1$

or

$a_n=[\![(n+1)\sqrt{2}]\!]-[\![n\sqrt{2}]\!]=[\![2+n\sqrt{2}]\!]-[\![n\sqrt{2}]\!]=2$

QED

Show that the sequence never contains three consecutive 1's or two consecutive 2's.

First we will show that the sequence never contains three consecutive 1's. For this we will let

$a_p = 1$ and $a_{p+1} = 1$. Then,

$a_p = [\![(p+1)\sqrt{2}]\!] - [\![p\sqrt{2}]\!] = 1$ and $a_{p+1} = [\![(p+2)\sqrt{2}]\!] - [\![(p+1)\sqrt{2}]\!] = 1$

By previous task, we note that since $a_p = 1$, then $[\![(p+1)\sqrt{2}]\!] = [\![1 + p\sqrt{2}]\!]$.

Therefore, since $a_{p+1} = 1$, then $[\![(p+2)\sqrt{2}]\!] = [\![2 + p\sqrt{2}]\!]$.

Now we must show that $a_{p+2} = 2$.

$$
\begin{aligned}
a_{p+2} &= [\![(p+3)\sqrt{2}]\!] - [\![(p+2)\sqrt{2}]\!] \\
&= [\![p\sqrt{2} + 3\sqrt{2}]\!] - [\![(p+2)\sqrt{2}]\!] \text{ by expanding the first,} \\
&= [\![p\sqrt{2} + 3\sqrt{2}]\!] - [\![2 + p\sqrt{2}]\!] \text{ by substituting for the second.}
\end{aligned}
$$

By calculation, we note that $4 < 3\sqrt{2} < 5$. Therefore, $p\sqrt{2} + 4 < p\sqrt{2} + 3\sqrt{2} < p\sqrt{2} + 5$.

Then $[\![p\sqrt{2} + 4]\!] \leq [\![p\sqrt{2} + 3\sqrt{2}]\!] \leq [\![p\sqrt{2} + 5]\!]$.

Show that the sequence never contains three consecutive 1's or two consecutive 2's.

Now we can see that

$[\![p\sqrt{2}+5]\!] - [\![p\sqrt{2}+4]\!] = [\![p\sqrt{2}]\!] + 5 - ([\![p\sqrt{2}]\!] + 4) = [\![p\sqrt{2}]\!] + 5 - [\![p\sqrt{2}]\!] - 4 = 5 - 4 =$

1.

Since $[\![p\sqrt{2}+3\sqrt{2}]\!] \in \mathbb{Z}$ and $[\![p\sqrt{2}+5]\!] - [\![p\sqrt{2}+4]\!] = 1$ and

$[\![p\sqrt{2}+4]\!] \leq [\![p\sqrt{2}+3\sqrt{2}]\!] \leq [\![p\sqrt{2}+5]\!]$,

$[\![p\sqrt{2}+3\sqrt{2}]\!] = [\![p\sqrt{2}+4]\!]$ or $[\![p\sqrt{2}+3\sqrt{2}]\!] = [\![p\sqrt{2}+5]\!]$.

Suppose $[\![p\sqrt{2}+3\sqrt{2}]\!] = [\![p\sqrt{2}+5]\!]$. Then

$a_{p+2} = [\![p\sqrt{2}+5]\!] - [\![2+p\sqrt{2}]\!] = [\![p\sqrt{2}]\!] + 5 - 2 - [\![p\sqrt{2}]\!] = 5 - 2 = 3.$

However, by the previous task, $a_{p+2} \neq 3$. Therefore $[\![p\sqrt{2}+3\sqrt{2}]\!] \neq [\![p\sqrt{2}+5]\!]$.

Thus, $[\![p\sqrt{2}+3\sqrt{2}]\!] = [\![p\sqrt{2}+4]\!]$.

Show that the sequence never contains three consecutive 1's or two consecutive 2's.

Then

$a_{p+2} = [\![p\sqrt{2} + 4]\!] - [\![2 + p\sqrt{2}]\!] = [\![p\sqrt{2}]\!] + 4 - 2 - [\![p\sqrt{2}]\!] = 4 - 2 = 2.$

Therefore $a_{p+2} = 2$.

Therefore if $a_p = 1$, and $a_{p+1} = 1$, then $a_{p+2} = 2$. Therefore there are not three consecutive one's in the sequence.

Now we will show that the sequence does not contain two consecutive 2's.

Let $a_m = 2$. Then we will show that $a_{m+1} = 1$.

$a_m = [\![(m+1)\sqrt{2}]\!] - [\![m\sqrt{2}]\!]$. From our previous task we note that $[\![(m+1)\sqrt{2}]\!] = [\![2 + m\sqrt{2}]\!]$.

Then $a_{m+1} = [\![(m+2)\sqrt{2}]\!] - [\![(m+1)\sqrt{2}]\!]$

$$= [\![(m+2)\sqrt{2}]\!] - [\![2 + m\sqrt{2}]\!]$$

$$= [\![m\sqrt{2} + 2\sqrt{2}]\!] - [\![2 + m\sqrt{2}]\!].$$

By calculation, we note that, $2 < 2\sqrt{2} < 3$. So, $m\sqrt{2} + 2 < m\sqrt{2} + 2\sqrt{2} < m\sqrt{2} + 3$.

So, $[\![m\sqrt{2} + 2]\!] \leq [\![m\sqrt{2} + 2\sqrt{2}]\!] \leq [\![m\sqrt{2} + 3]\!]$.

Show that the sequence never contains three consecutive 1's or two consecutive 2's.

Now we can see that

$\llbracket m\sqrt{2}+3 \rrbracket - \llbracket m\sqrt{2}+2 \rrbracket = \llbracket m\sqrt{2} \rrbracket + 3 - (\llbracket m\sqrt{2} \rrbracket + 2) = \llbracket m\sqrt{2} \rrbracket + 3 - \llbracket m\sqrt{2} \rrbracket - 2 = 3 -$

$2 = 1.$

Since $\llbracket m\sqrt{2}+2\sqrt{2} \rrbracket \epsilon \mathbb{Z}$ and $\llbracket m\sqrt{2}+3 \rrbracket - \llbracket m\sqrt{2}+2 \rrbracket = 1$ and

$\llbracket m\sqrt{2}+2 \rrbracket \leq \llbracket m\sqrt{2}+2\sqrt{2} \rrbracket \leq \llbracket m\sqrt{2}+3 \rrbracket,$

$\llbracket m\sqrt{2}+2\sqrt{2} \rrbracket = \llbracket m\sqrt{2}+2 \rrbracket$ or $\llbracket m\sqrt{2}+2\sqrt{2} \rrbracket = \llbracket m\sqrt{2}+3 \rrbracket.$

Suppose that $\llbracket m\sqrt{2}+2\sqrt{2} \rrbracket = \llbracket m\sqrt{2}+2 \rrbracket$. Then,

$a_{m+1} = \llbracket m\sqrt{2}+2 \rrbracket - \llbracket m\sqrt{2}+2 \rrbracket = \llbracket m\sqrt{2} \rrbracket + 2 - \llbracket m\sqrt{2} \rrbracket - 2 = 2 - 2 = 0.$

Show that the sequence never contains three consecutive 1's or two consecutive 2's.

However, by the previous task, we know that $a_{m+1} \neq 0$. Therefore, $[\![m\sqrt{2} + 2\sqrt{2}]\!] \neq [\![m\sqrt{2} + 2]\!]$.

Thus, $[\![m\sqrt{2} + 2\sqrt{2}]\!] = [\![m\sqrt{2} + 3]\!]$. Then,

$a_{m+1} = [\![m\sqrt{2} + 2\sqrt{2}]\!] - [\![m\sqrt{2} + \sqrt{2}]\!] = [\![m\sqrt{2} + 3]\!] - [\![m\sqrt{2} + 2]\!] = [\![m\sqrt{2}]\!] + 3 - [\![m\sqrt{2}]\!] - 2 = 3 - 2 = 1$.

Therefore, $a_{m+1} = 1$.

Therefore if $a_m = 2$, then $a_{m+1} = 1$. Therefore there are not two consecutive two's in the sequence.

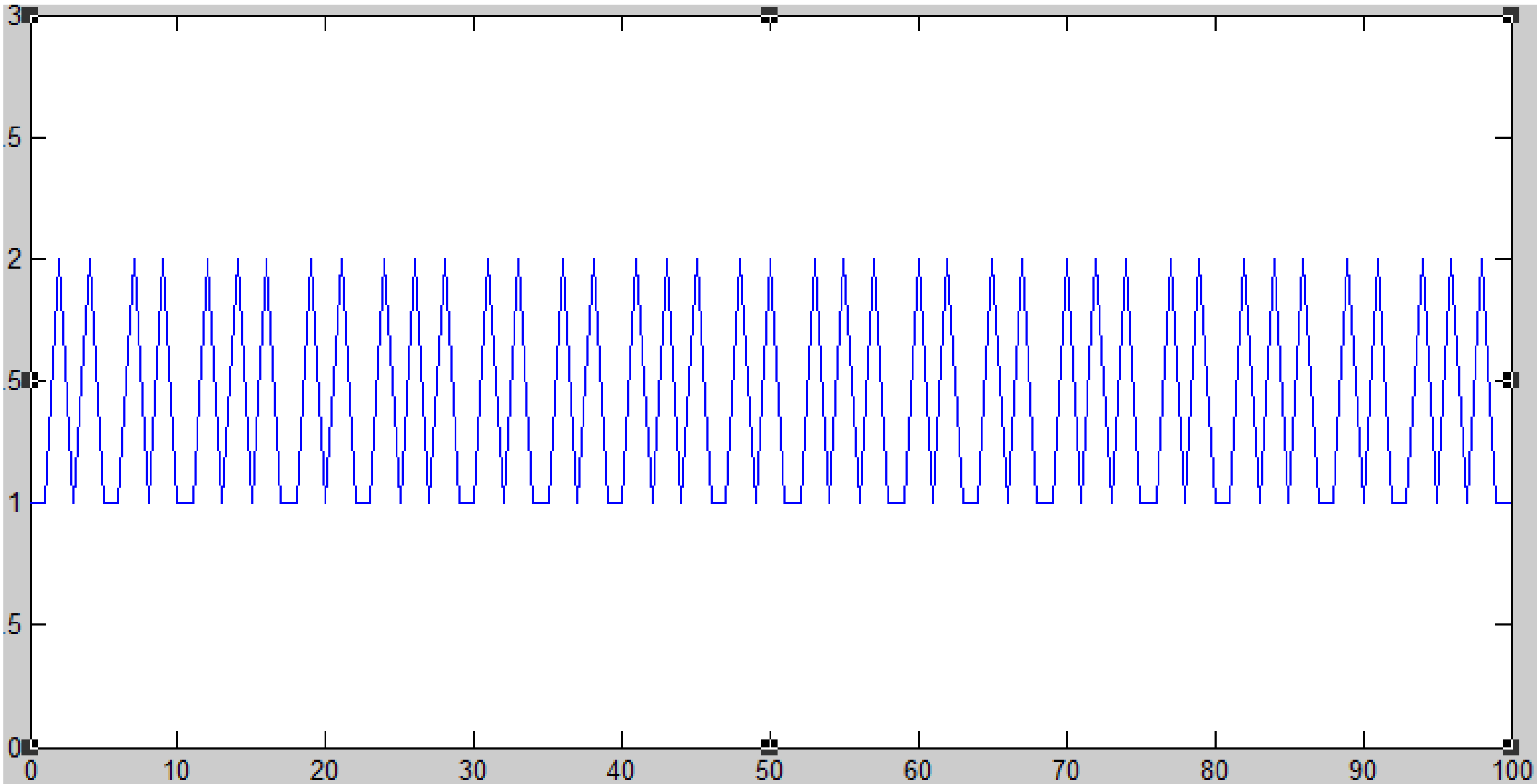
DWG

Describe all k-term subsequences that occur , for k=3,4,…

- >> n=[0:100];
- >> n=floor((n+1).*sqrt(2))-floor(n.*sqrt(2))
- n = 1 1 2 1 2 1 1 2 1 2 1 1
  2 1 2 1 2 1 1 2 1 2 1 1 2
  1 2 1 2 1 1 2 1 2 1 1 2 1
  2 1 1 2 1 2 1 2 1 1 2 1 2
  1 1 2 1 2 1 2 1 2 1 2 1 1
  2 1 2 1 1 2 1 2 1 2 1 1 2
  1 2 1 1 2 1 2 1 2 1 1 2 1
  2 1 1 2 1 2 1 2 1 1

Describe all k-term subsequences that occur , for k=3,4,…

plot(floor((n+1).*sqrt(2))-floor(n.*sqrt(2)))



Describe all k-term subsequences that occur , for k=3,4,…

k	WORKS	DOESN'T WORK
3	(1,1,2); (1,2,1); (2,1,1); (2,1,2)	(1,1,1); (2,2,2); (1,2,2); (2,2,1)
4	(1,1,2,1); (1,2,1,1); (1,2,1,2); (2,1,2,1); (2,1,1,2)	(1,1,1,1); (2,2,2,1); (1,2,2,2); (2,2,2,2); (2,1,1,1); (1,1,1,2)
5	(1,1,2,1,2); (1,2,1,2,1); (2,1,2,1,1); (1,2,1,1,2); (2,1,1,2,1); (2,1,2,1,2);	(1,1,1,1,1); (1,2,2,2,1); (1,1,2,2,2); (1,2,2,2,2); (1,2,1,1,1); (1,1,1,1,2); (2,1,1,1,1); (2,2,2,2,1); (2,1,2,2,2); (2,2,2,2,2); (2,2,1,1,1); (2,1,1,1,2); (2,2,2,1,1); (1,2,2,2,1); (2,2,2,2,1); (1,1,1,2,1) **(1,1,2,1,1)**
6	(1,1,2,1,2,1); (1,2,1,2,1,1); (1,2,1,2,1,2); (2,1,1,2,1,2); (2,1,2,1,1,2); (2,1,2,1,2,1); (1,1,2,1,2,1); (1,2,1,1,2,1)	All sequences with less than two 2's or more than three 2's. Also with the conditions from the previous task applied. Including (**1,1,2,1,1**,2); (2,**1,1,2,1,1**)
7	(1,1,2,1,2,1,1); (2,1,1,2,1,2,1); (1,1,2,1,2,1,2);	All sequences without three 2's and with the

A general pattern in the sequence and proof of sum of subsequences

We observed that for any subsequence of length *n*, $\frac{\sum_{i=k}^{k+(n-1)} a(i)}{n}$ is the closest fraction to $\sqrt{2}$ with *n* as a denominator. This means that the average of any subsequence of length *n* is the closest fraction to $\sqrt{2}$ with *n* as a denominator. We will prove this by showing that given an *n* length subsequence, the sum of the *n*-terms is equal to $\lfloor n\sqrt{2} \rfloor \; or \; \lfloor n\sqrt{2} \rfloor + 1$.

Proof of sum of subsequences

Proof: Let $b \in \mathbb{Z}$. Then

$a_b = \lfloor (b+1)\sqrt{2} \rfloor - \lfloor b\sqrt{2} \rfloor$

$a_{b+1} = \lfloor (b+2)\sqrt{2} \rfloor - \lfloor (b+1)\sqrt{2} \rfloor$

$a_{b+2} = \lfloor (b+3)\sqrt{2} \rfloor - \lfloor (b+2)\sqrt{2} \rfloor$

...

$a_{b+n-3} = \lfloor (b+n-2)\sqrt{2} \rfloor - \lfloor (b+n-3)\sqrt{2} \rfloor$

$a_{b+n-2} = \lfloor (b+n-1)\sqrt{2} \rfloor - \lfloor (b+n-2)\sqrt{2} \rfloor$

$a_{b+n-1} = \lfloor (b+n)\sqrt{2} \rfloor - \lfloor (b+n-1)\sqrt{2} \rfloor$

Then $\sum_{i=b}^{b+n-1} a_i = \lfloor (b+1)\sqrt{2} \rfloor - \lfloor b\sqrt{2} \rfloor + \lfloor (b+2)\sqrt{2} \rfloor - \lfloor (b+1)\sqrt{2} \rfloor + \lfloor (b+3)\sqrt{2} \rfloor - \lfloor (b+2)\sqrt{2} \rfloor + \cdots + \lfloor (b+n-2)\sqrt{2} \rfloor - \lfloor (b+n-3)\sqrt{2} \rfloor + \lfloor (b+n-1)\sqrt{2} \rfloor - \lfloor (b+n-2)\sqrt{2} \rfloor + \lfloor (b+n)\sqrt{2} \rfloor - \lfloor (b+n-1)\sqrt{2} \rfloor$.

Notice that all of the coordinating colored expressions cancel out.

So, $\sum_{i=b}^{b+n-1} a_i = \lfloor (b+n)\sqrt{2} \rfloor - \lfloor b\sqrt{2} \rfloor$.

We now note that i) $\lfloor n\sqrt{2} \rfloor \le n\sqrt{2} \le \lfloor n\sqrt{2} \rfloor + 1 \Rightarrow$

$\lfloor b\sqrt{2} + \lfloor n\sqrt{2} \rfloor \rfloor \le \lfloor b\sqrt{2} + n\sqrt{2} \rfloor \le \lfloor b\sqrt{2} + \lfloor n\sqrt{2} \rfloor \rfloor + 1$

ii) $\left| \lfloor b\sqrt{2} + (\lfloor n\sqrt{2} \rfloor + 1) \rfloor - \lfloor b\sqrt{2} + \lfloor n\sqrt{2} \rfloor \rfloor \right| = 1$

iii) $\lfloor b\sqrt{2} + n\sqrt{2} \rfloor \in \mathbb{Z}$

From the noted above, we conclude that

$\lfloor b\sqrt{2} + n\sqrt{2} \rfloor = \lfloor b\sqrt{2} + \lfloor n\sqrt{2} \rfloor \rfloor$ or $\lfloor b\sqrt{2} + n\sqrt{2} \rfloor = \lfloor b\sqrt{2} + \lfloor n\sqrt{2} \rfloor \rfloor + 1$

So,

$\sum_{i=b}^{b+n-1} a_i = \lfloor b\sqrt{2} + \lfloor n\sqrt{2} \rfloor \rfloor - \lfloor b\sqrt{2} \rfloor$ or $= \lfloor b\sqrt{2} + \lfloor n\sqrt{2} \rfloor \rfloor + 1 - \lfloor b\sqrt{2} \rfloor$

Since $\lfloor n\sqrt{2} \rfloor$ and $\lfloor n\sqrt{2} \rfloor + 1 \in \mathbb{Z}$, then they can be pulled out of the floor function. So,

Proof of sum of subsequences

$\sum_{i=b}^{b+n-1} a_i = \lfloor b\sqrt{2} \rfloor + \lfloor n\sqrt{2} \rfloor - \lfloor b\sqrt{2} \rfloor$ or $= \lfloor b\sqrt{2} \rfloor + \lfloor n\sqrt{2} \rfloor + 1 - \lfloor b\sqrt{2} \rfloor$

$= \lfloor n\sqrt{2} \rfloor$ $= \lfloor n\sqrt{2} \rfloor + 1.$

QED

Proof that this shows for any subsequence of length n, $\frac{\sum_{i=k}^{k+(n-1)} a(i)}{n}$ is the closest fraction to $\sqrt{2}$ with n as a denominator:

Again we note i) $\lfloor n\sqrt{2} \rfloor \le n\sqrt{2} \le \lfloor n\sqrt{2} \rfloor + 1$

=>

$\frac{\lfloor n\sqrt{2} \rfloor}{n} \le \frac{n\sqrt{2}}{n} \le \frac{\lfloor n\sqrt{2} \rfloor + 1}{n}$ =>

$\frac{\lfloor n\sqrt{2} \rfloor}{n} \le \sqrt{2} \le \frac{\lfloor n\sqrt{2} \rfloor + 1}{n}$

Since $(\lfloor n\sqrt{2} \rfloor + 1) - \lfloor n\sqrt{2} \rfloor = 1$, then $\nexists$ an integer, x, s.t. $\lfloor n\sqrt{2} \rfloor < x < \lfloor n\sqrt{2} \rfloor + 1$. Therefore, $\nexists$ a fraction, $\frac{c}{n}$, such that $\frac{\lfloor n\sqrt{2} \rfloor}{n} < \frac{c}{n} < \frac{\lfloor n\sqrt{2} \rfloor + 1}{n}$. Therefore, $\frac{\lfloor n\sqrt{2} \rfloor}{n}$ is the closest fraction less than $\sqrt{2}$ and $\frac{\lfloor n\sqrt{2} \rfloor + 1}{n}$ is the closest fraction greater than $\sqrt{2}$.

QED

Therefore, we are able to find the nth term in the sequence if we know the (n-1) terms and observe the patterns in the subsequences prior to n.

One can replace $\sqrt{2}$ by any real number. Conjecture and prove patterns for other numbers.

Integers:

If we replace $\sqrt{2}$ with a, where $a \in \mathbf{Z}$, then the sequence corresponding to $\lfloor (n+1)a \rfloor - \lfloor na \rfloor$ is the sequence

$a, a, a, a, a, a, a, a, a, a, a, a, a, a, a, \ldots.$

This is because $\lfloor (n+1)a \rfloor \in \mathbf{Z}$ and $\lfloor na \rfloor \in \mathbf{Z}$,

so $\lfloor (n+1)a \rfloor - \lfloor na \rfloor = (n+1)a - na = na + a - na = a$.

Rationals:

If we replace $\sqrt{2}$ with b, where $b \in \mathbf{Q}$, then the sequence corresponding to $\lfloor (n+1)b \rfloor - \lfloor nb \rfloor$ is a binary sequence consisting of $\lfloor b \rfloor, \lceil b \rceil$. The sequence is defined by

$$b_n = \begin{cases} \lfloor b \rfloor, & \text{if } (nx)\ mod\ y \le x\ mod\ y \\ \lceil b \rceil, & \text{if } (nx)\ mod\ y > x\ mod\ y \end{cases}$$ where $x =$ numerator, $y =$ denominator

Ex) $b = \frac{3}{7}$

$b_n = \{0, 1, 0, 1, 0, 1, 0, 0, 1, 0 \ldots\}$

To show this we will put it in a table:

n	**x**	**nx**	**x mod y**	**nx mod y**	**b_n =**
1	3	3	3	3	0
2	3	6	3	6	1
3	3	9	3	2	0
4	3	12	3	5	1
5	3	15	3	1	0
6	3	18	3	4	1
7	3	21	3	0	0

The sequence will repeat this subsequence since the sequence is defined modulo 7.

Any Questions?