# An Arithmetic Sum

Project: Explore the following arithmetic function.

$$For\ m > 0, let\ f(m) = \sum_{r=1}^{m} \frac{m}{\gcd(m,r)}.$$

We determined the first step of a complete solution as follows:

Since a prime is only divisible by one and itself, the gcd would be 1 on the interval (1, p-1), and p for the final additive. The value of $f(p)$ is the sum (p-1) added a total of p times, plus one [the final element of the series will always be one, since r will be equal to m, leading to the gcd equaling p in value and canceling].

$$f(p) = p(p-1) + 1 = p^2 - p + 1$$

Examples

$f(1) = \frac{1}{1} = 1$ $f(2) = \frac{2}{1} + \frac{2}{2} = 3$ $f(3) = \frac{3}{1} + \frac{3}{1} + \frac{3}{3} = 7$

$f(4) = \frac{4}{1} + \frac{4}{2} + \frac{4}{1} + \frac{4}{4} = 11$

$f(5) = \frac{5}{1} + \frac{5}{1} + \frac{5}{1} + \frac{5}{1} + \frac{5}{5} = 21$

$f(6) = \frac{6}{1} + \frac{6}{2} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} + \frac{6}{6} = 21$

$f(10) = \frac{10}{1} + \frac{10}{2} + \frac{10}{1} + \frac{10}{2} + \frac{10}{5} + \frac{10}{2} + \frac{10}{1} + \frac{10}{2} + \frac{10}{1} + \frac{10}{10} = 63$

$f(12) = \frac{12}{1} + \frac{12}{2} + \frac{12}{3} + \frac{12}{4} + \frac{12}{1} + \frac{12}{6} + \frac{12}{1} + \frac{12}{4} + \frac{12}{3} + \frac{12}{2} + \frac{12}{1} + \frac{12}{12} = 77$

For m is prime power $p^2$, $p^2$ is added together p-1 times inside the brackets, until reaching a multiple of prime p. There are p multiples of p in $p^2$ by definition of "squared," so there are p instances of p-1 long stretches of $p^2$, giving our first term $p^2(p-1)(p)$. For every multiple of prime p except for $p^2$, the corresponding term in the sequence is equal to p; this occurs a total of p-1 times, leading to the second term p*(p-1).

$$f(p^2) = [p^2 + p^2 + \cdots + p^2 + p^2] + p + [p^2 + p^2 + \cdots + p^2 + p^2] + p + \cdots + [p^2 + p^2 + \cdots + p^2 + p^2] + 1 = p^2(p-1)(p) + [p(p-1) + 1]$$

Or, factoring p out of the (p-1) $[p^2 + p^2 + \ldots + p^2 + p]$ terms, we get (p-1)*(p)*(p+p+p+p+... +1), and by substitution $f(p^2) = (p-1)p * f(p) + p^2 + p^2 + \ldots + p^2 + 1$. Then, rewrite $p^2 + p^2 + \ldots + p^2 + 1$ as $[p^2 + p^2 + \ldots + p^2 + p] - p + 1$, which is $p * f(p) - p + 1$

Finally, $f(p^2) = (p-1)p*f(p) + p*f(p) - p + 1 = p^2*f(p) - p + 1 = p^4 - p^3 + p^2 - p + 1$

<u>Proof by Induction</u>: Let's assume, according to the pattern for the formulas of f(p) and $f(p^2)$ that $f(p^k) = p^{2k} - p^{2k-1} + p^{2k-2} - p^{2k-3} + \ldots - p + 1$. Then, using the same process as we did in $f(p^2)$

$f(p^{k+1}) = (p-1)p*f(p^k) + p^{k+1} + p^{k+1} + \ldots + p^{k+1} + 1 = (p-1)p*f(p^k) + p^{k+1} + p^{k+1} + \ldots + p^{k+1} + p - p + 1 = (p-1)p*f(p^k) + p*f(p^k) - p + 1 = p^2*f(p^k) - p + 1$
$= p^{2k+2} - p^{2k+1} + p^{2k} - p^{2k-1} + \ldots - p + 1$

**Thus, $f(p^n) = p^{2n} - p^{2n-1} + p^{2n-2} - p^{2n-3} + \ldots - p + 1$ for all positive integers n.**